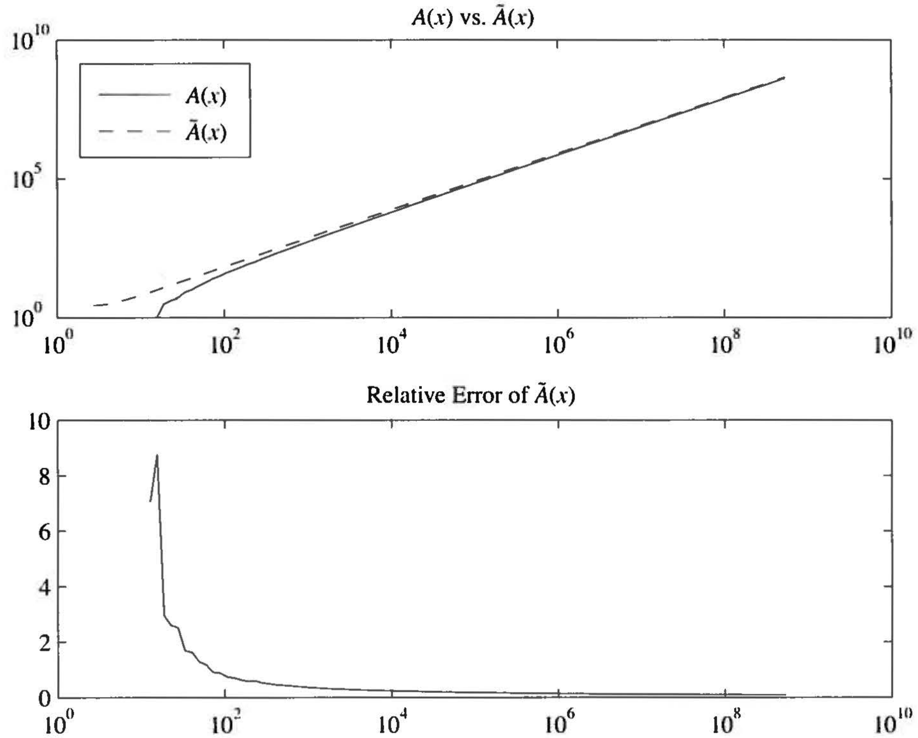

FIG. 4. *Comparison of the function $A(x)$ and its estimate function $\tilde{A}(x)$ of number of multiplicatively abundant numbers*

[10] ERIC WEISSTEIN, Eric Weisstein's World of Mathematics, http://mathworld.wolfram.com/

William Chau, SoftTechies Corp., 24 Bucknell Dr., E. Brunswick, NJ 08816. wchau@softtechies.com

William Chau obtained his undergraduate degrees in Mathematics, Computer Science, and Electrical Engineering at SUNY at Buffalo. During the following ten years working as a software developer, he continued his postgraduate study to receive two master degrees, the first in Computer Science from Stevens Institute of Technology, and the second in Mathematics from the Courant Institute at NYU. He recently founded SoftTechies Corp., and has been doing independent software consulting ever since.

IIME Journal, Vol. 11, No. 10, pp 535–538, 2004.

EXTREMAL INTERSECTION SIZES

VALERIO DE ANGELIS* AND ALLEN STENGER

1. Introduction. Suppose we are investigating a finite collection of finite sets, and suppose we know the size of each set and the size of their union. What, if anything, can we say about the size of their intersection? We write n for the number of sets, and A_i for the sets. If there are only two sets, then the exact size of the intersection is given by

$$|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|. \tag{1}$$

However, when $n > 2$ we don't have enough information to determine the size of the intersection. In this note we will generalize the two-set case to get tight upper and lower bounds on the size of the intersection. The inspiration for this note is a problem that we received at MathNerds.com [2]; we solve this problem in Section 2 as an application of the results in this note. A similar problem was presented by Lewis Carroll in [3, Knot X, §1, pp. 142–144], and we discuss his solutions in Section 4.

THEOREM 1. *With the notation above,*

$$\sum_i |A_i| - (n-1)\left|\bigcup_i A_i\right| \le \left|\bigcap_i A_i\right| \le \frac{1}{n-1}\left(\sum_i |A_i| - \left|\bigcup_i A_i\right|\right).$$

For $n = 2$, the upper and lower bounds are the same and we get equation (1). Note that for $n > 2$, the bounds given by Theorem 1 may be worse than the trivial bounds $0 \le |\cap_i A_i| \le \min_i |A_i|$. The next theorem shows that Theorem 1 combined with the trivial bounds gives the best-possible bounds. If c is in the range specified by (3) and (4) below, then there is a collection having intersection size c, and if c is not in that range then Theorem 1 and the trivial bounds show that no collection can have intersection size c.

THEOREM 2. *Suppose each of $u, a_1, \ldots, a_n$ is a positive integer, and suppose*

$$\max_i a_i \le u \le \sum_i a_i. \tag{2}$$

Suppose c is an integer satisfying

$$0 \le c \le \min_i a_i \tag{3}$$

and

$$\sum_i a_i - (n-1)u \le c \le \frac{1}{n-1}\left(\sum_i a_i - u\right). \tag{4}$$

Then there is a collection of sets $\{A_i\}$ having sizes $|A_i| = a_i$, union size $|\cup_i A_i| = u$, and intersection size $|\cap_i A_i| = c$.

*Xavier University of Louisiana

2. Example. A health club has 600 members. Of these, 80% are male, 70% play tennis, and 60% swim. What are the lowest and highest possible percentages of males who swim and play tennis?

Let A_1, A_2, A_3 be the sets of the 480 members who are male, the 420 who play tennis, and the 360 who swim, respectively. The intersection of these sets is the set of males who swim and play tennis. The size of the union of these three sets is at most 600, but we don't know the exact size because there may be some women who neither swim nor play tennis. Therefore Theorems 1 and 2 do not apply directly, but we do have estimates for the union size and we can use these in the theorems. Write $u = |\cup_i A_i|$ and $c = |\cap_i A_i|$. Theorem 1 tells us $480 + 420 + 360 - 2u \le c \le \frac{1}{2}(480 + 420 + 360 - u)$. From the sizes of A_i and the total membership we know $480 \le u \le 600$. Using these upper and lower bounds for u we get $60 \le c \le 390$. The upper bound is worse than the trivial bound 360. Our conclusion is that, for any value of u, we have $60 \le |\cap_i A_i| \le 360$.

We don't know yet that these extremal values can be met for a total membership of 600, but we will show this by construction. The idea is to put aside all members of the intersection in one group, and then spread out the properties of the remaining members as much as possible, such that none of the remaining members are in all the sets. (We'll use the same idea in general form to prove Theorem 2.)

For the lower bound, first assign 60 of the members to be male, play tennis, and swim. Then we have 540 members remaining, from which we will assign properties such that 420 are male, 360 are tennis players, and 300 are swimmers, and such that no person is all three. Imagine arranging the remaining 540 members in a big circle, and numbering them counterclockwise 1–540. Go around the circle assigning properties as follows: Members 1–420 are male, members 421–540 and 1–240 play tennis, and members 241–540 swim. Then none of these 540 is male, plays tennis, and swims.

The upper bound construction is similar. First assign 360 of the members to be male, play tennis, and swim. Then we have 240 members remaining, from which we need 120 males, 60 tennis players, and 0 swimmers, such that no person is all three. Arrange the remaining members in a circle and number them 1–240, and assign their properties as follows: Members 1–120 are male, members 121–180 play tennis, and no members swim. Then none of these 240 is male, plays tennis, and swims.

3. Proofs. Proof of Theorem 1: We use a counting argument. Write $U = \cup_i A_i$ and write C_k for the set of elements of U that are in exactly k of the A_i. Then $\sum_k |C_k| = |U|$ and $C_n = \cap_i A_i$. The sum $\sum_i |A_i|$ counts each element of C_k exactly k times, so $\sum_i |A_i| = \sum_k k|C_k|$. We can now estimate

$$\sum_i |A_i| = \sum_{k=1}^{n-1} k|C_k| + n|C_n| \le \sum_{k=1}^{n-1}(n-1)|C_k| + n|C_n| = (n-1)|U| + |C_n|$$

and solving the inequality gives $|\cap_i A_i| = |C_n| \ge \sum_i |A_i| - (n-1)|U|$.

A similar argument gives the upper bound:

$$\sum_i |A_i| = \sum_{k=1}^{n-1} k|C_k| + n|C_n| \ge \sum_{k=1}^{n-1} 1 \cdot |C_k| + n|C_n| = |U| + (n-1)|C_n|$$

and therefore $|C_n| \le (1/(n-1))(\sum_i |A_i| - |U|)$. □

Proof of Theorem 2: We construct the sets A_i by drawing two circles in the plane, cutting them into arcs of unit length, and defining the A_i as certain collections of these arcs. The size of a set is both the number of arcs in the set and their total length, and we use this duality in the proof.

If $c = 0$, let $C = \emptyset$, otherwise draw a circle of circumference c and divide it into c arcs of unit length. From inequalities (2) and (3) we have $u \ge \max_i a_i \ge \min_i a_i \ge c$. If $u = c$ then we can take all $A_i = C$, because then $|\cup_i A_i| = |\cap_i A_i| = |C| = c = u$. Therefore assume $u > c$ in the following.

Draw a second circle, disjoint from the first, and of circumference $u - c$. Divide the circle into $u - c$ arcs of unit length. Define the set B_1 as the set of the first $a_1 - c$ arcs, B_2 as the set of the next $a_2 - c$ arcs, and so on (some B_i may be empty). From inequality (2) we have that $a_i - c \le u - c$, and so no B_i wraps around to contain the same arc twice. Finally define $A_i = B_i \cup C$.

Because the circles are disjoint, they have no arcs in common and it is clear that $|A_i| = (a_i - c) + c = a_i$. We need to show that $|\cup_i A_i| = u$ and that $|\cap_i A_i| = c$. Because C is disjoint from each B_i, these are equivalent to showing $|\cup_i B_i| = u - c$ and that $\cap_i B_i = \emptyset$.

To show that $|\cup_i B_i| = u - c$, observe from inequality (4) that $\sum_i a_i \ge (n-1)c + u$ and so $\sum_i (a_i - c) = \sum_i a_i - nc \ge (n-1)c + u - nc = u - c$. Therefore the sum of the lengths of the arcs in the B_i is at least $u - c$, the circumference of the circle. The arcs in the B_i are contiguous, so $\cup_i B_i$ contains all arcs of the circle.

To show that $\cap_i B_i = \emptyset$, from the left-hand inequality of (4) we have that $\sum_i a_i \le c + (n-1)u$ and therefore $\sum_i (a_i - c) = \sum_i a_i - nc \le c + (n-1)u - nc = (n-1)(u-c)$. Therefore the B_i may wrap around the circle, but not more than $n - 1$ times, and so there are no arcs that are in more than $n - 1$ of the B_i. Therefore $\cap_i B_i = \emptyset$. □

4. Related results. We believe the results in this note must be already known, but we were not able to find them anywhere in the literature. The branch of combinatorics that deals with this kind of problem is Extremal Set Theory; see for example [1, Chapter 6].

A related result is the Inclusion-Exclusion Principle that is discussed in most books on combinatorics, for example, [1, Chapter 10]. It states in our notation that if $V \supseteq U$ then

$$|V \setminus U| = |V| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |\cap_i A_i|.$$

This is another generalization of equation (1), because we can take $V = U$ and $n = 2$ to get equation (1). The Inclusion-Exclusion Principle is proved by another counting argument. An element of $V \setminus U$ is counted exactly once by the right-hand side because it appears only in V, and an element of U that is in exactly k of the A_i is counted

$$1 - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k \binom{k}{k} = (1-1)^k = 0$$

times.

The Inclusion-Exclusion Principle is used in many kinds of combinatorial problems and is also the basis of sieve methods in number theory, but it requires that you know or can estimate the sizes of all possible combinations of intersections of the A_i, unlike Theorem 1 that only requires knowledge of the size of the sets and of their union.

We received the Example problem (lower bound only) at MathNerds.com [2], a web site offering free help to students. The problem was submitted by Ms. Regis R. Park and solved by MathNerds volunteer Esther Fontova by a "worst case" analysis, pushing as many of the swimmers and tennis players as possible into the female category, then pushing the remaining swimmers and tennis players as far apart as possible.

In [3, Knot X, §1, pp. 142–144], Lewis Carroll states a problem about the Chelsea Pensioners (war veterans):

> If 70 per cent have lost a eye, 75 per cent an ear, 80 per cent an arm, and 85 per cent a leg: what percentage, *at least*, must have lost all four?

The problems in [3] were published as a serial in a monthly magazine, and Carroll published and discussed in later issues the solutions sent in by the readers. The solution to the Chelsea Pensioners that he judged best used a version of the Pigeonhole Principle: Suppose we have 100 men, and add up all the injuries, so that we have 70 lost eyes and so on, for a total of 70 + 75 + 80 + 85 = 310 injuries among 100 men, so at least 10 men must have all four injuries. Carroll also presents briefly a second solution, which is essentially to spread out the injuries as much as possible among the 100 men, leading to the least overlap.

Carroll's solutions are discussed in more generality and in more mathematical terms in Herstein & Kaplansky [4, pp. 12–14]. They treat the first solution as an example of De Morgan's law. We seek the size of the intersection of the four classes of men, and by De Morgan the complement of this set is the union of the complements. The size of the union is bounded above by the sum of the sizes of the complements, or 30 + 25 + 20 + 15 = 90, so the size of the original intersection is bounded below by 10. They also treat Carroll's second solution by three applications of (1).

Both Carroll and Herstein & Kaplansky gloss over the problem of whether the lower bound is actually reached (although in this example it is easy to show that it is). Neither discussion attempts to investigate the upper bound (which would be the trivial upper bound 70 in this example) or to generalize the problem.

REFERENCES

[1] VAN LINT, J. H. and WILSON, R. M., "A Course in Combinatorics", 2nd ed., Cambridge University Press, 2001.

[2] DE ANGELIS, VALERIO; DAWKINS, PAUL; MAHAVIER, W. TED; and STENGER, ALLEN, *MathNerds Offers Discovery-Style Mathematics on the Web,* FOCUS, Vol. 22, No. 2, pp. 10–11. February 2002.

[3] CARROLL, LEWIS, "A Tangled Tale", in: "Pillow Problems and A Tangled Tale", Dover reprint, 1958.

[4] HERSTEIN, I. N. and KAPLANSKY, I., "Matters Mathematical", 2nd ed., Chelsea, 1978.

Valerio De Angelis, Mathematics Department, Xavier University of Louisiana, 1 Drexel Drive, New Orleans, LA 70125. vdeangel@xula.edu

Allen Stenger, 29 Cielo Montana, Alamogordo, NM 88310. StenMathMail@aol.com

Valerio De Angelis is an Assistant Professor of Mathematics at Xavier University of Louisiana in New Orleans, and a co-founder of MathNerds. His interests include Dynamical Systems and Asymptotic Analysis.

Allen Stenger is a retired software developer, a math hobbyist, and a volunteer at MathNerds.com. His mathematical interests are in number theory and classical analysis.

IIME Journal, Vol. 11, No. 10, pp 539–544, 2004.

HEROIC TETRAHEDRA AND SIMPLEXES

I. J. GOOD*

The famous formula, named after Hero (or Heron) of Alexandria and sometimes attributed to Archimedes, for the area Δ of a triangle, is

$$\Delta^2 = s(s-a)(s-b)(s-c), \tag{1}$$

where the lengths of the sides are denoted by a, b, c and $s = \frac{1}{2}(a+b+c)$. Hersh (2002) gave elegant methods for deriving (1) from the assumption that Δ^2 is a polynomial in the lengths of the edges (and therefore has to be a quartic). Without a proof, we don't know that Δ^2 isn't, for example, the square root of a polynomial of the eighth degree. One way to prove that Δ^2 is indeed a polynomial is to use Theorem 4036 in Carr (1970). He gives a simple geometrical proof of a formula for Δ in terms of the coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ of the vertices. If we take the third vertex as the origin O, then that formula reduces to one half of the absolute value of $x_1y_2 - x_2y_1$ and this also is a well-known formula. It can be written, if preferred, in terms of a 2 by 2 determinant

$$2\Delta = \pm \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \tag{2}$$

On multiplying this determinant by its transpose, we get the symmetric determinant

$$4\Delta^2 = \begin{vmatrix} x_1^2 + y_1^2 & x_1x_2 + y_1y_2 \\ x_1x_2 + y_1y_2 & x_2^2 + y_2^2 \end{vmatrix} = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix} \tag{3}$$

The elements of the right-hand determinant, which *no longer depend on the location of the origin*, are inner or scalar products, where the bold **a** and **b** denote the sides, a and b, expressed more fully as vectors (that is, with their *directions* represented). To obtain a formula in terms of the lengths of the sides alone write $\mathbf{a}.\mathbf{b}$ as $1/2(a^2+b^2-c^2)$ (even when $a = b$, in which case $c = 0$), and we obtain

$$16\Delta^2 = \begin{vmatrix} 2a^2 & a^2+b^2-c^2 \\ a^2+b^2-c^2 & 2b^2 \end{vmatrix}. \tag{4}$$

This is a polynomial in a, b, and c, as is required to convert Hersh's derivation of Hero's formula into a proof. But in a sense we have proved too much, for the determinant equals (by the factorization formula for the "difference of two squares" applied thrice)

$$\begin{aligned} 4a^2b^2 &- (a^2+b^2-c^2)^2 && (5)\\ &= [2ab + (a^2+b^2-c^2)][2ab - (a^2+b^2-c^2)] \\ &= [(a+b)^2 - c^2][c^3 - (a-b)^2] \\ &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) && (6) \end{aligned}$$

and this proves Hero's formula (1).

Thus formula (4) leads to a proof of Hero's formula by algebra even more elementary than what Hersh used in his 'derivations'. But I like the didactic aspects of Hersh's methods.

*Virginia Tech